*

X \ Y	2	1	0
2	0.2	0.1	0
1	0.1	0.2	0.1
0	0	0.1	0.2

(X, Y)	prob.
(2, 2)	0.2
(2, 1)	0.1
(1, 2)	0.1
(1, 1)	0.2
(1, 0)	0.1
(0, 1)	0.1
(0, 0)	0.2

indicator r.v.

$$\frac{E(Y\,I(X=2))}{P(X=2)}$$

$\rightarrow$ Is $X=2$? — If it is $=1$, if it is not $=0$

① $E(Y|X) = ?$

$$E(Y|X=2) = \frac{2\cdot 1\cdot 0.2 + 1\cdot 1\cdot 0.1 + \underbrace{2\cdot 0\cdot 0.1}_{=0} + \dots}{0.3} = \frac{5}{3}$$

$$E(Y|X=1) = \frac{E(Y\,I(X=1))}{P(X=1)} = \frac{2\cdot 0\cdot 0.2 + 1\cdot 0\cdot 0.1 + 2(1)(0.1) + 1(1)(0.2) + 0(1)(0.1)}{0.4} = \frac{0.4}{0.4} = 1$$

$$E(Y|X=0) = \frac{1}{3}$$

You could also write it as

$$E(Y|X=x) = \begin{cases} 5/3 & \text{if } x = 2 \\ 1 & \text{if } x = 1 \\ 1/3 & \text{if } x = 0 \end{cases}$$

$$P(Y=2|X=2) = \frac{P(Y=2\ \&\ X=2)}{P(X=2)} = 0.2/0.3 = 2/3$$

(2) BLP of Y given X $\qquad \beta_0^* + \beta_1^* X$

$$\beta_0^* = \mathbb{E}(Y) - \beta_1^* \mathbb{E}(X)$$

$$\beta_1^* = \frac{Cov(X,Y)}{Var(X)}$$

$$= \mathbb{E}((X-\mathbb{E}(X))(Y-\mathbb{E}(Y)))$$

$$Cov(X,Y) = \boxed{\mathbb{E}(XY) - \mathbb{E}(X)\boxed{\mathbb{E}(Y)}}$$

$$= (2\cdot 2\cdot 0.2 + 2\cdot 1\cdot 0.1 + \cdots)$$

$$- [2\cdot 0.3 + 1\cdot 0.4 + 0\cdot 0.3]$$

$$[2\cdot 0.3 + 1\cdot 0.4 + 0\cdot 0.3]$$

$$= \bigcirc$$

(X, Y)	prob.
(2, 2)	0.2
(2, 1)	0.1
(1, 2)	0.1
(1, 1)	0.2
(1, 0)	0.1
(0, 1)	0.1
(0, 0)	0.2

$$\mathbb{E}(Y|X=x) = \begin{cases} 5/3 & \text{if } x = 2 \\ 1 & \text{if } x = 1 \\ 1/3 & \text{if } x = 0 \end{cases}$$

* Alternatively:

$$\mathbb{E}(XY) = \mathbb{E}(\mathbb{E}(\boxed{X}Y|X))$$

law of iterated expectations.

I condition on X because I know $\mathbb{E}(Y|X)$

prop. of cond. exp $= \mathbb{E}(\boxed{X}\,\boxed{\mathbb{E}(Y|X)})$

also a function of X

$$= 2\,\mathbb{E}(Y|X=2)\cdot 0.3$$
$$+ 1\,\mathbb{E}(Y|X=1)\cdot 0.4$$
$$+ 0\,\mathbb{E}(Y|X=0)\cdot 0.3 = 1.4$$

law of iterated exp. $\quad \mathbb{E}(Y) = \mathbb{E}\big(\underbrace{\mathbb{E}(Y|X)}_{\text{function of } X}\big)$

$$= \frac{5}{3} \cdot 0.3 + 1 \cdot 0.4 + \frac{1}{3} \cdot 0.3$$
$$= 1.$$

After the really excruciating work $\Rightarrow$ $\quad \beta_0^* = \frac{1}{3}, \quad \beta_1^* = \frac{2}{3}.$

BLP of Y given X $= \frac{1}{3} + \frac{2}{3}X$

NOTE

$$\mathbb{E}(Y|X=x) = \begin{cases} 5/3 & \text{if } x = 2 \\ 1 & \text{if } x = 1 \\ 1/3 & \text{if } x = 0 \end{cases}$$

$\to$ BLP of Y given X $= \frac{1}{3} + \frac{2}{3}X$

Here CEF = BLP!

③ Derive the dist of $\varepsilon = Y - \mathbb{E}(Y|X)$. $\to$ function of X

(X, Y)	prob.	$\varepsilon = Y - \mathbb{E}(Y\mid X)$
$(2, 2)$	0.2	$2 - 5/3 = 1/3$
$(2, 1)$	0.1	$1 - 5/3 = -2/3$
$(1, 2)$	0.1	$2 - 1 = 1$
$(1, 1)$	0.2	$1 - 1 = 0$
$(1, 0)$	0.1	$0 - 1 = -1$
$(0, 1)$	0.1	$1 - 1/3 = 2/3$
$(0, 0)$	0.2	$0 - 1/3 = -1/3$

$\mathbb{P}(\varepsilon = \frac{1}{3}) = 0.2$

$\mathbb{P}(\varepsilon = -2/3) = 0.1$

$\vdots$

④ Is $\mathbb{E}(X^2 \varepsilon) = 0$?

First way: $\mathbb{E}(X^2 \varepsilon) = 2^2 \cdot \frac{1}{3} \cdot 0.2 + 2^2 \cdot \left(-\frac{2}{3}\right)(0.1) + \cdots$

$= 0$

Second way: $\mathbb{E}(X^2 \varepsilon) = \mathbb{E}\left(\mathbb{E}(X^2 \varepsilon \mid X)\right)$ law of iterated exp.

$\overset{?}{=} \mathbb{E}\left(X^2\, \mathbb{E}(\varepsilon \mid X)\right) = 0.$

Find $\mathbb{E}(\varepsilon \mid X) = \mathbb{E}\left(Y - \mathbb{E}(Y\mid X) \mid X\right)$

$= \mathbb{E}(Y\mid X) - \mathbb{E}\left(\boxed{\mathbb{E}(Y\mid X)} \mid X\right)$

function of X

$= \mathbb{E}(Y|X) - \mathbb{E}(Y|X) = 0.$

$$\mathbb{E}(\overset{\text{r.v.}}{Y}) = \mathbb{E}(\mathbb{E}(\overset{\text{r.v.}}{Y}|\overset{\text{r.v.}}{X}))$$

$$= \mathbb{E}(\underbrace{\mathbb{E}(Y|Y)}_{Y})$$

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|Y))$$

could not use this for the exercise!

TE 2 $\mathbb{E}(Y|X) = X^2$

(1)

- $\mathbb{E}(Y|X=2) = 4$ Is this true or false?

  $\mathbb{E}(Y|X=2) = 2^2 = 4.$ TRUE!

- $\mathbb{E}(Y) = 4$? Is this true or false?

No choice but to use law of iterated expectations. Why? You do not know the joint dist of (X,Y)!

$\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|X)) = \mathbb{E}(X^2) \overset{\text{given in exercise}}{=} 1.$ FALSE!

- $\mathbb{E}(\varepsilon | X=2) = 2$? Is this true or false?

  $\varepsilon = Y - \mathbb{E}(Y|X) \Rightarrow \mathbb{E}(\varepsilon|X=2) = \mathbb{E}(Y - \mathbb{E}(Y|X) | X=2)$

$$\mathbb{E}(\underbrace{\mathbb{E}(Y|X)}_{\text{known / constant}}|X) = \mathbb{E}(Y|X), \quad X=2 \;\rightarrow\; X=2$$

$$\mathbb{E}(Y|X)=X^2 \;\rightarrow\; = \mathbb{E}(Y|X=2) - \mathbb{E}(\underbrace{\mathbb{E}(Y|X)}_{\text{function of } X}|X=2)$$

$$= 4 - \mathbb{E}(Y|X=2)$$

$$= 4 - 4 = 0$$

FALSE!

(2) BLP of Y given X?

$$\beta_0^* + \beta_1^* X = 1 + 0\cdot X$$

$$\beta_0^* = \mathbb{E}(Y) - \beta_1^*\mathbb{E}(X), \quad \beta_1^* = \frac{Cov(X,Y)}{Var(X)}$$

You do not see a table similar to TE1! So $Cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$ Stuck!

$\mathbb{E}(\boxed{2}X)$

Aronow & Miller example (slides)

$$Cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

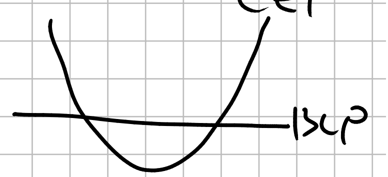
$$\rightarrow \mathbb{E}(\mathbb{E}(\boxed{X}Y|X)) \qquad \mathbb{E}(\mathbb{E}(Y|X))$$

$$= \mathbb{E}(X\,\mathbb{E}(Y|X)) - \mathbb{E}(X)\,\mathbb{E}(X^2)$$

$$= \mathbb{E}(X^3) - \mathbb{E}(X)\mathbb{E}(X^2) = 0 - 0\cdot 1 = 0.$$

$$\Rightarrow \beta_1^* = 0$$

$$\beta_0^* = \mathbb{E}(Y) = 1.$$



(3) $\hat\beta_0 \xrightarrow{p} 1, \quad \hat\beta_1 \xrightarrow{p} 0$ BLP of Y given X

(4) Note that $\mathbb{E}(Y|X) = 1\cdot X^2$ (given at the beginning)

N

$$E(Y|X) = \boxed{\beta_0^* + \beta_1^* X + \beta_2^* X^2}$$

where $\beta_0^* = 0$

$\beta_1^* = 0$

$\beta_2^* = 1$

CEF is BLP

BLP of Y given X, X^2

$\underset{\substack{\shortparallel \\ X}}{X1}$, $\underset{\substack{\shortparallel \\ X^2}}{X2}$

lm(Y ~ X1 + X2) $\rightarrow$ $\hat{\beta}_0 \xrightarrow{p} 0$, $\hat{\beta}_1 \xrightarrow{p} 0$, $\hat{\beta}_2 \xrightarrow{p} 1$.

BLP of Y given X, X^2, X^3 with intercept

NOTE TE2 is special because you can express CEF as a linear predictor.

Here $E(Y|X) = \frac{1}{X+2}$ cannot be expressed as $\beta_0^* + \beta_1^* X + \dots$

$E(Y|X) = \log X$ $\rightarrow$ BLP of Y given X $\rightarrow$ difficult $\beta_0^* + \beta_1^* X$

but BLP of Y given $\log X$ $\rightarrow$ $\beta_0^* + \beta_1^* \log X$

TE 3

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + v$$

A1 $\quad E(v \mid X_1, X_2) = 0$

✓ A2 $\quad E(v \mid X_1, X_2) = E(v \mid X_2) = \delta_0 + \delta_1 X_2$ for simplicity

↳ knowing X_1 does not help in predicting v if you already conditioned on X_2.

(conditional mean independence)

(1)

Without A1 or A2, $\beta_0 + \beta_1 X_1 + \beta_2 X_2$ is not a CEF of Y given X_1 + X_2!

Why? $E(Y \mid X_1, X_2) \underset{\text{eqn.}}{=} E(\beta_0 + \beta_1 X_1 + \beta_2 X_2 + v \mid X_1, X_2)$

$$= E(\beta_0 \mid X_1, X_2) + E(\beta_1 X_1 \mid X_1, X_2) + E(\beta_2 X_2 \mid X_1, X_2) + E(v \mid X_1, X_2)$$

$$= \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \underbrace{E(v \mid X_1, X_2)}_{=0 \text{ by A1}}$$

(2) By A1, $E(Y \mid X_1, X_2) \overset{(1)}{=} \beta_0 + \beta_1 X_1 + \beta_2 X_2 + 0$

$$= \beta_0 + \beta_1 X_1 + \beta_2 X_2$$

Yes!

$\beta_0^* \quad \beta_1^* \quad \beta_2^*$

(3)

NOTE $Y = \beta_0 + \beta_1 X_1 + v$ A1 $E(v|X_1) = 0$

Under A1, $\beta_0 = \beta_0^*$, $\beta_1 = \beta_1^*$.

$\beta_0^* = E(Y) - \beta_1^* E(X_1)$ $\beta_1^* = \frac{Cov(X_1, Y)}{Var(X_1)} = \ldots = \beta_1$

NOTE $E(v|X_1, X_2) = 0 \Rightarrow E(v) = 0$ law of iterated exp.

$E(v) = E(E(v|X_1, X_2))$

$\Rightarrow E(v|X_1, X_2) = E(v) = 0$ compare with A2!

(4)

$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + v$

From (1), $E(Y|X_1, X_2) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + E(v|X_1, X_2)$

$\overset{A2}{=} \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \delta_0 + \delta_1 X_2$

$= (\beta_0 + \delta_0) + \beta_1 X_1 + \boxed{(\beta_2 + \delta_1)} X_2$

So, NO, $\boxed{\beta_0} + \beta_1 X_1 + \boxed{\beta_2} X_2$ is not the CEF of Y given X_1 + X_2!

(5) $E(Y \mid X_1 = x_1 + 1, X_2 = x_2) - E(Y \mid X_1 = x_1, X_2 = x_2)$

under A1	under A2
$\beta_0 + \beta_1(x_1+1) + \beta_2 x_2 - [\beta_0 + \beta_1 x_1 + \beta_2 x_2]$	$((\beta_0+\delta_0) + \beta_1(x_1+1) + (\beta_2+\delta_1)x_2 - [(\beta_0+\delta_0) + \beta_1 x_1 + (\beta_2+\delta_1)x_2]$
$= \beta_1$ //	$= \beta_1$ //

$E(Y \mid X_1 = x_1, X_2 = \underline{x_2 + 1}) - E(Y \mid X_1 = x_1, X_2 = \underline{x_2})$

under A1	under A2
$\beta_0 + \beta_1 x_1 + \beta_2(x_2+1) - [\beta_0 + \beta_1 x_1 + \beta_2 x_2]$	$(\beta_0+\delta_0) + \beta_1 x_1 + (\beta_2+\delta_1)(x_2+1) - [(\beta_0+\delta_0) + \beta_1 x_1 + (\beta_2+\delta_1) x_2]$
$= \beta_2$ /	$= \beta_2 \boxed{+\delta_1}$ $\neq 0$

$\hat\beta_0, \hat\beta_1, \hat\beta_2$ OLS lm()

(6) under $\underline{A1}$ $\quad \hat\beta_0 \xrightarrow{p} \beta_0^* \; (= \beta_0)$, $\quad \boxed{\hat\beta_1 \xrightarrow{p} \beta_1^* \; (= \beta_1)}$, $\quad \hat\beta_2 \xrightarrow{p} \beta_2^* \; (= \beta_2)$

under $\underline{A2}$ $\quad \hat\beta_0 \xrightarrow{p} \beta_0 + \delta_0$, $\quad \boxed{\hat\beta_1 \xrightarrow{p} \beta_1}$, $\quad \hat\beta_2 \xrightarrow{p} \beta_2 + \delta_1$

applied work → on the search for "good" control variables.

$Y \quad X_1 \quad \boxed{X_2} =$

variable of interest / focus variable

control variables / doubtful variables