Technical Exercise 1

(3) X_1 and X_2 are independent r.v.'s because the two draws are made with replacement.
(Note: Explanations using the setup of the problem are adequate. Actually showing independence based on the definition you have on the slides takes more work. In the discrete case, we can check whether $\mathbb{P}(X_1 = a \text{ and } X_2 = b) = \mathbb{P}(X_1 = a) * \mathbb{P}(X_2 = b)$ for all a, b. In the continuous case, it will involve densities.)

(5) One way is to draw without replacement, meaning we can draw one ball and do not return it to the container.
There are other ways, for example, whenever $X_1 = 2$, we automatically set $X_2 = 0$.

Items 1, 2, 4 are available as an Excel file at the website.

Technical Exercise 2

①

$$
\begin{aligned}
\mathrm{Var}(aX+bY+c) &\overset{\text{def of variance}}{=} \mathbb{E}\left[(aX+bY+c-\mathbb{E}(aX+bY+c))^2\right] \\
&\overset{\text{properties of the expectation}}{=} \mathbb{E}\left[(aX+bY+c-a\mathbb{E}(X)-b\mathbb{E}(Y)-c)^2\right] \\
&= \mathbb{E}\left[(a(X-\mathbb{E}(X))+b(Y-\mathbb{E}(Y)))^2\right] \\
&= \mathbb{E}\left[a^2(X-\mathbb{E}(X))^2+2ab(X-\mathbb{E}(X))(Y-\mathbb{E}(Y))+b^2(Y-\mathbb{E}(Y))^2\right] \\
&\overset{\text{properties of the expectation}}{=} a^2\,\mathbb{E}\left[(X-\mathbb{E}(X))^2\right]+2ab\,\mathbb{E}\left[(X-\mathbb{E}(X))(Y-\mathbb{E}(Y))\right]+b^2\,\mathbb{E}\left[(Y-\mathbb{E}(Y))^2\right] \\
&\overset{\text{def of variance, covariance}}{=} a^2\,\mathrm{Var}(X)+2ab\,\mathrm{Cov}(X,Y)+b^2\,\mathrm{Var}(Y)
\end{aligned}
$$

algebraic simplification/grouping

NOTE An alternative is to use $\mathrm{Var}(X)=\mathbb{E}(X^2)-\mathbb{E}(X)^2$ as a starting point.

②

$$
\begin{aligned}
\mathrm{Var}(\underset{\uparrow\, a}{2}X\underbrace{-3}_{\uparrow\, b}Y\underbrace{+4}_{\uparrow\, c}) &= 4\,\mathrm{Var}(X)+2(2)(-3)\,\mathrm{Cov}(X,Y)+9\,\mathrm{Var}(Y) \\
&= 4(3-0^2)-12(0.5)\sqrt{3\cdot 2}+9(3-1^2) \\
&= 30-6\sqrt{6} \approx 15.3
\end{aligned}
$$

NOTE $\rho_{X,Y}=\dfrac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} \Rightarrow \mathrm{Cov}(X,Y)=\rho_{X,Y}\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}$

Technical Exercise 3

(1) $\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$

def of covariance

$= \mathbb{E}(X \cdot X^2) - \mathbb{E}(X)\mathbb{E}(X^2)$

def. of Y

$= \mathbb{E}(X^3) - \mathbb{E}(X)\mathbb{E}(X^2)$

moments of standard normal

$= 0 - 0(1) = 0$

(2) $\mathrm{Cov}(X^2, Y) = \mathbb{E}(X^2 \cdot X^2) - \mathbb{E}(X^2)\mathbb{E}(X^2)$

$= \mathbb{E}(X^4) - \mathbb{E}(X^2)\mathbb{E}(X^2)$

$= 3 - 1 \cdot 1 = 2 \neq 0$

(3) You can use (2) to show that Y and X are not independent.
But seeing that $\mathrm{Cov}(X,Y) = 0$ is not enough to show independence or lack of independence!

From the definition, we can show that X & Y are dependent because we can find functions H_1 and H_2 such that

$$\mathbb{E}(H_1(X)\,H_2(Y)) \neq \mathbb{E}(H_1(X))\,\mathbb{E}(H_2(Y))$$

Take $H_1(X) = X^2$, $H_2(Y) = Y$. So $\mathbb{E}(X^2 Y) \neq \mathbb{E}(X^2)\mathbb{E}(Y)$ from (2).