Technical Exercise 1

① Let $y = \log_b x$. Then $b^y = x$. Taking logarithms (base a) of both sides, we have $\log_a b^y = \log_a x$. From the properties of logarithms, we have $y \log_a b = \log_a x$. Solving for y gives $y = \dfrac{\log_a x}{\log_a b}$.

Therefore, $\log_b x = \dfrac{\log_a x}{\log_a b}$.

② Consider the following mappings:

$b \leftarrow e$, $a \leftarrow 10$, $x \leftarrow$ Net Sales, $y \leftarrow$ TotalComp

$X_1 \leftarrow \log_a x$, $Y \leftarrow \log_a y$, $W \leftarrow \log_b x$, $Z \leftarrow \log_b y$

Because of ①, we have $W_t = \frac{1}{\log_a b} X_{1t}$ & $Z_t = \frac{1}{\log_a b} Y_t$.

From Technical Exercise 2 of Ex Set 03, we know that

slope of regression of Z on W $= \dfrac{1/\log_a b}{1/\log_a b}$ * slope of regression of Y on X_1.

Therefore, the regression slopes are the same.

Technical Exercise 2

① $p = \log x_2 - \log x_1 \iff p = \log\left(\frac{x_2}{x_1}\right)$

$$\iff e^p = \frac{x_2}{x_1}$$

$$\iff e^p - 1 = \frac{x_2 - x_1}{x_1}$$

$$\iff (e^p - 1) * 100\% = \left(\frac{x_2 - x_1}{x_1}\right) * 100\%$$

②

p	$e^p - 1$
0	0
0.01	≈ 0.01
0.1	≈ 0.105
0.2	≈ 0.221
0.3	≈ 0.350

p	$e^p - 1$
0.4	≈ 0.492
0.5	≈ 0.649
0.6	≈ 0.822
0.7	≈ 1.014
0.8	≈ 1.226

p	$e^p - 1$
0.9	≈ 1.46
1	≈ 1.718

The approximation is quite severe as soon as $p > 0.3$. the error starts from about 5 percentage points to 72 percentage points!

Technical Exercise 3

Case 1. Regression with only an intercept

Here $X_t = 1$. Thus,

$$\hat{\beta} = \left(\frac{1}{n}\sum_{t=1}^{n} X_t X_t'\right)^{-1}\left(\frac{1}{n}\sum_{t=1}^{n} X_t Y_t\right) = \left(\frac{1}{n}\sum_{t=1}^{n} 1\cdot 1\right)^{-1}\left(\frac{1}{n}\sum_{t=1}^{n} 1\cdot Y_t\right)$$

$$= \left(\frac{1}{n}\cdot n\right)^{-1}\left(\frac{1}{n}\sum_{t=1}^{n} Y_t\right) = \overline{Y}.$$

Case 2. Simple linear regression

Here $X_t = (1, X_{1t})'$. Thus,

$$\hat{\beta} = \left(\frac{1}{n}\sum_{t=1}^{n} X_t X_t'\right)^{-1}\left(\frac{1}{n}\sum_{t=1}^{n} X_t Y_t\right) = \left(\frac{1}{n}\sum_{t=1}^{n}\begin{pmatrix}1\\ X_{1t}\end{pmatrix}(1 \ X_{1t})\right)^{-1}\left(\frac{1}{n}\sum_{t=1}^{n}\begin{pmatrix}1\\ X_{1t}\end{pmatrix} Y_t\right)$$

$$= \left(\frac{1}{n}\sum_{t=1}^{n}\begin{pmatrix}1 & X_{1t}\\ X_{1t} & X_{1t}^2\end{pmatrix}\right)^{-1}\left(\frac{1}{n}\sum_{t=1}^{n}\begin{pmatrix}Y_t\\ X_{1t}Y_t\end{pmatrix}\right)$$

$$= \begin{pmatrix}\frac{1}{n}\sum_{t=1}^{n} 1 & \frac{1}{n}\sum_{t=1}^{n} X_{1t}\\ \frac{1}{n}\sum_{t=1}^{n} X_{1t} & \frac{1}{n}\sum_{t=1}^{n} X_{1t}^2\end{pmatrix}^{-1}\begin{pmatrix}\frac{1}{n}\sum_{t=1}^{n} Y_t\\ \frac{1}{n}\sum_{t=1}^{n} X_{1t}Y_t\end{pmatrix}$$

$$= \begin{pmatrix} 1 & \overline{X}_1 \\ \overline{X}_1 & \frac{1}{n}\sum_{t=1}^{n} X_{1t}^2 \end{pmatrix}^{-1} \begin{pmatrix} \overline{Y} \\ \frac{1}{n}\sum_{t=1}^{n} X_{1t} Y_t \end{pmatrix}$$

$$= \frac{1}{\frac{1}{n}\sum_{t=1}^{n} X_{1t}^2 - (\overline{X}_1)^2} \begin{pmatrix} \frac{1}{n}\sum_{t=1}^{n} X_{1t}^2 & -\overline{X}_1 \\ -\overline{X}_1 & 1 \end{pmatrix} \begin{pmatrix} \overline{Y} \\ \frac{1}{n}\sum_{t=1}^{n} X_{1t} Y_t \end{pmatrix}$$

$$= \frac{1}{\frac{1}{n}\sum_{t=1}^{n} X_{1t}^2 - (\overline{X}_1)^2} \begin{pmatrix} \left(\frac{1}{n}\sum_{t=1}^{n} X_{1t}^2\right)\overline{Y} - \overline{X}_1 \left(\frac{1}{n}\sum_{t=1}^{n} X_{1t} Y_t\right) \\ -\overline{X}_1 \overline{Y} + \frac{1}{n}\sum_{t=1}^{n} X_{1t} Y_t \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\left(\frac{1}{n}\sum_{t=1}^{n} X_{1t}^2\right)\overline{Y} - \overline{X}_1 \left(\frac{1}{n}\sum_{t=1}^{n} X_{1t} Y_t\right)}{\frac{1}{n}\sum_{t=1}^{n} X_{1t}^2 - (\overline{X}_1)^2} \\ \dfrac{\frac{1}{n}\sum_{t=1}^{n} X_{1t} Y_t - \overline{X}_1 \overline{Y}}{\frac{1}{n}\sum_{t=1}^{n} X_{1t}^2 - (\overline{X}_1)^2} \end{pmatrix}$$

From past exercises, $\frac{1}{n}\sum_{t=1}^{n} X_{1t} Y_t - \overline{X}_1 \overline{Y} =$

$\frac{1}{n}\sum_{t=1}^{n} X_{1t}^2 - (\overline{X}_1)^2 =$

Therefore, $\dfrac{\frac{1}{n}\sum_{t=1}^{n} X_{1t}Y_t - \bar{X}_1\bar{Y}}{\frac{1}{n}\sum_{t=1}^{n} X_{1t}^2 - (\bar{X}_1)^2} = \dfrac{\frac{1}{n}\sum_{t=1}^{n}(X_{1t}-\bar{X}_1)(Y_t-\bar{Y})}{\frac{1}{n}\sum_{t=1}^{n}(X_{1t}-\bar{X}_1)^2} = \hat{\beta}_1$.

$$\frac{\left(\frac{1}{n}\sum_{t=1}^{n} X_{1t}^2\right)\bar{Y} - \bar{X}_1\left(\frac{1}{n}\sum_{t=1}^{n} X_{1t}Y_t\right)}{\frac{1}{n}\sum_{t=1}^{n} X_{1t}^2 - (\bar{X}_1)^2} = \frac{\left(\frac{1}{n}\sum_{t=1}^{n} X_{1t}^2\right)\bar{Y} - (\bar{X}_1)^2\bar{Y} + (\bar{X}_1)^2\bar{Y} - \bar{X}_1\left(\frac{1}{n}\sum_{t=1}^{n} X_{1t}Y_t\right)}{\frac{1}{n}\sum_{t=1}^{n} X_{1t}^2 - (\bar{X}_1)^2}$$

$$= \frac{\bar{Y}\left(\frac{1}{n}\sum_{t=1}^{n} X_{1t}^2 - (\bar{X}_1)^2\right) - \bar{X}_1\left[\frac{1}{n}\sum_{t=1}^{n} X_{1t}Y_t - \bar{X}_1\bar{Y}\right]}{\frac{1}{n}\sum_{t=1}^{n} X_{1t}^2 - (\bar{X}_1)^2}$$

$$= \bar{Y} - \bar{X}_1 \cdot \hat{\beta}_1 = \hat{\beta}_0$$

Technical Exercise 4

1

$$\frac{\partial}{\partial \hat\beta_0} \sum_{t=1}^{n} \left(Y_t - \hat\beta_0 - \hat\beta_1 X_{1t}\right)^2$$

$$= \sum_{t=1}^{n} \frac{\partial}{\partial \hat\beta_0} \left(Y_t - \hat\beta_0 - \hat\beta_1 X_{1t}\right)^2$$

$$= \sum_{t=1}^{n} 2\left(Y_t - \hat\beta_0 - \hat\beta_1 X_{1t}\right)(-1)$$

$$\frac{\partial}{\partial \hat\beta_1} \sum_{t=1}^{n} \left(Y_t - \hat\beta_0 - \hat\beta_1 X_{1t}\right)^2$$

$$= \sum_{t=1}^{n} \frac{\partial}{\partial \hat\beta_1} \left(Y_t - \hat\beta_0 - \hat\beta_1 X_{1t}\right)^2$$

$$= \sum_{t=1}^{n} 2\left(Y_t - \hat\beta_0 - \hat\beta_1 X_{1t}\right)(-X_{1t})$$

The first order conditions are

$$\begin{cases} \sum_{t=1}^{n} (-2)\left(Y_t - \hat\beta_0 - \hat\beta_1 X_{1t}\right) = 0 \\ \sum_{t=1}^{n} (-2X_{1t})\left(Y_t - \hat\beta_0 - \hat\beta_1 X_{1t}\right) = 0 \end{cases}$$

properties of summation $\Rightarrow$

$$\begin{cases} -2\sum_{t=1}^{n} \left(Y_t - \hat\beta_0 - \hat\beta_1 X_{1t}\right) = 0 \\ -2\sum_{t=1}^{n} X_{1t}\left(Y_t - \hat\beta_0 - \hat\beta_1 X_{1t}\right) = 0 \end{cases}$$

divide both sides by (-2) $\Rightarrow$

$$\begin{cases} \sum_{t=1}^{n} \left(Y_t - \hat\beta_0 - \hat\beta_1 X_{1t}\right) = 0 \\ \sum_{t=1}^{n} X_{1t}\left(Y_t - \hat\beta_0 - \hat\beta_1 X_{1t}\right) = 0 \end{cases}$$

(2)

$$\sum_{t=1}^{n} (Y_t - \hat{\beta}_0 - \hat{\beta}_1 X_{1t}) = 0 \Rightarrow \sum_{t=1}^{n} Y_t - n\hat{\beta}_0 - \hat{\beta}_1 \sum_{t=1}^{n} X_{1t} = 0$$

$$\Rightarrow \sum_{t=1}^{n} Y_t = n\hat{\beta}_0 + \hat{\beta}_1 \sum_{t=1}^{n} X_{1t}$$

$$\sum_{t=1}^{n} X_{1t}(Y_t - \hat{\beta}_0 - \hat{\beta}_1 X_{1t}) = 0 \Rightarrow \sum_{t=1}^{n} X_{1t} Y_t - \hat{\beta}_0 \sum_{t=1}^{n} X_{1t} - \hat{\beta}_1 \sum_{t=1}^{n} X_{1t}^2 = 0$$

$$\Rightarrow \sum_{t=1}^{n} X_{1t} Y_t = \hat{\beta}_0 \sum_{t=1}^{n} X_{1t} + \hat{\beta}_1 \sum_{t=1}^{n} X_{1t}^2$$

(3)

$$\begin{pmatrix} n & \sum_{t=1}^{n} X_{1t} \\ \sum_{t=1}^{n} X_{1t} & \sum_{t=1}^{n} X_{1t}^2 \end{pmatrix} \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \begin{pmatrix} \sum_{t=1}^{n} Y_t \\ \sum_{t=1}^{n} X_{1t} Y_t \end{pmatrix}$$

(4)

(a) $$\sum_{t=1}^{n} \underbrace{(Y_t - \hat{\beta}_0 - \hat{\beta}_1 X_{1t})} = 0 \Rightarrow \sum_{t=1}^{n} e_t = 0$$

this is the residual for the t^{th} observation. Optimal $(\hat{\beta}_0, \hat{\beta}_1)$ have to satisfy first-order conditions!

Start from:

(b) correlation coefficient between X_1 and residuals

$$= \frac{\frac{1}{n}\sum_{t=1}^{n}(X_{1t}-\bar{X}_1)(e_t-\bar{e})}{\sqrt{\frac{1}{n}\sum_{t=1}^{n}(X_{1t}-\bar{X}_1)^2}\sqrt{\frac{1}{n}\sum_{t=1}^{n}(e_t-\bar{e})^2}}$$

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Note that $\frac{1}{n}\sum_{t=1}^{n}(X_{1t}-\bar{X}_1)(e_t-\bar{e}) = \frac{1}{n}\sum_{t=1}^{n}X_{1t}e_t - \bar{X}_1\bar{e}$

$\bar{e}=0$ from (a)

$$= \frac{1}{n}\sum_{t=1}^{n}X_{1t}e_t$$

definition of residual

$$= \frac{1}{n}\sum_{t=1}^{n}X_{1t}(Y_t-\hat{\beta}_0-\hat{\beta}_1X_{1t})$$

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$$= 0$$

Therefore, correlation coefficient between X_1 and residuals is zero.

(c) From $\sum_{t=1}^{n}(Y_t-\hat{\beta}_0-\hat{\beta}_1X_{1t})=0$, we have $\sum_{t=1}^{n}Y_t = \sum_{t=1}^{n}(\hat{\beta}_0+\hat{\beta}_1X_{1t})$

But $\hat{Y}_t=\hat{\beta}_0+\hat{\beta}_1X_{1t}$, so $\sum_{t=1}^{n}Y_t=\sum_{t=1}^{n}\hat{Y}_t$. Therefore $\frac{1}{n}\sum_{t=1}^{n}Y_t=\frac{1}{n}\sum_{t=1}^{n}\hat{Y}_t$.